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SOME RESULTS ON UNIFORMLY LIPSCHITZIAN MAPPINGS IN METRIC SPACES AND APPLICATIONS

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Abstract. In this work, we obtain a useful property of uniformly Lipschitzian mappings, then we give some suitable conditions under which uniformly Lipschitzian mappings have a fixed point without limiting conditions on the Lipschitz constant L in metric spaces. As an application of our main results we investigate the solvability of Fredholm integral equations of second kind. Some illustrative examples are included to show the usefulness and applicability of results.

Keywords: fixed-point theorems, uniformly Lipschitzian mappings, nonexpansive mappings, Fredholm integral equations.

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1. Introduction and preliminaries

Let (X, d) be a metric space and F be a self map on X. We say that:

(1) F is nonexpansive if for all $x, y \in X$ we have
$$d(Fx, Fy) \leqslant d(x, y).$$

(2) F is uniformly α-Lipschitzian if there exists constant $\alpha > 0$ such that for all $x, y \in X$ and $n \in \mathbb{N}$ we have
$$d(F^n x, F^n y) \leqslant \alpha d(x, y). \tag{1}$$

The smallest α for which (1) holds is said to be the Lipschitz constant for F and is denoted by L.

Fixed point theory has an abundance of applications in proving the existence of solutions for a wide class of differential and integral equations, see [3, 4, 5, 13, 27] and the references cited therein.

In recent years, the existence of fixed point for a uniformly Lipschitzian mapping (for short ULM) has been considered extensively not only in uniformly convex Banach spaces but even in metric spaces (see, eg, [18, 20] and references cited therein).

ULMs are generalization of the nonexpansive mappings, in fact a nonexpansive map is a uniformly 1-Lipschitzian map. Furthermore, some problems in the stability are closely related to the study of the existence of fixed points for ULMs.

In 1973, ULMs in uniformly convex Banach spaces was initiated by Goebel and Kirk [9] which proved a fixed point theorem in normed spaces by using the concept of modulus of convexity. Then some authors studied the existence of a fixed point for ULMs under suitable conditions in CAT(0) spaces [1].

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In 1989, Khamsi [18] introduced the concept of normal and uniform normal structures for metric spaces, then proved a fixed point theorem for nonexpansive mappings that has been extended by others [20, 23, 26].

As it is common, some authors introduced a new class of ULMs called generalized ULMs [15, 16]. Also, some authors worked on the structure of fixed-point sets of ULMs in Banach spaces [6, 11, 22] also, in 2009, Jarosław Górnicki [12] extended and improved some of previous results.

Recently, one of the celebrated results about the existence of a fixed point for ULMs has been proved in Hilbert spaces as

Theorem 1.1. [2] *Let H be a Hilbert space and C be a nonempty bounded closed convex subset of H. If $T : C \to C$ is a uniformly L-Lipschitzian mapping with $L < \sqrt{2}$, then T has a fixed point in C.*

In recent years, we can still find different types of extensions such as, multivalued ULMs ([17]) and uniformly L-Lipschitzian asymptotically pseudocontractive mappings ([25]).

Let self map F on metric space (X, d) be a ULM. If we define the function ρ with

$$\rho(x, y) = \sup\Big\{d(F^n x, F^n y) : n = 0, 1, ...\Big\} \text{ for all } x, y \in X, \tag{2}$$

then it is easy to see that ρ is a metric on X that is equivalent to d and F is nonexpansive in (X, ρ). ([10])

Let (X, d) be a metric space. A map $F : X \to X$ is called compact if it maps every bounded sequence $\{x_n\}$ in X onto a sequence $\{F(x_n)\}$ in X which has a convergent subsequence.

The aim of this paper is to obtain some new fixed point results for ULMs in metric spaces, we do this in section 2. In sectin 3 some illustrative examples are given to demonstrate the effectiveness of the obtained results.

To conclude this section we mention the following classical result of Edelstein in metric spaces.

Theorem 1.2. [24] *Let (X, d) be a metric space and $F : X \to X$ be a nonexpansive mapping. If there is $z \in X$ such that the sequence $\{F^n z\}$ has a convergent subsequence to a point $x \in X$, then F is distance preserving on the set $\big\{F^n(x) : n = 0, 1, 2, ...\big\}$. In other word, for all nonnegative integers s, p, r we have*

$$d(F^{s+r}x, F^{p+r}x) = d(F^s x, F^p x).$$

2. Fixed point results

In the present section, we first obtain a property of ULMs regarding the asymptotic behavior of Picard iterations. Further, by using the obtained result, we establish some sufficient conditions under which a ULM has a fixed point.

We begin with the following simple lemma:

Lemma 2.1. *Let (X, d) be a metric space and $F : X \to X$ be a ULM such that the sequence $\{F^n z\}$ has a convergent subsequence for all $z \in X$. Then there exists an $x \in X$ such that*

$$\inf_{n \geqslant N} d(F^n x, x) = 0 \textit{ for all } N \in \mathbb{N}.$$

Proof. Consider the metric ρ on X as defined in (2). Since the map F is nonexpansive in the metric space (X, ρ), we can apply Theorem 1.2 to find $x \in X$ such that for all nonnegative integers s, p, r we have

$$\rho(F^{s+r}x, F^{p+r}x) = \rho(F^s x, F^p x).$$

Thus, for all $m, n \in \mathbb{N}$ with $m < n$ we have

$$\rho(F^n x, F^m x) = \rho(F^{n-m} x, x). \tag{3}$$

Let $\epsilon > 0$ and $N \in \mathbb{N}$ be arbitrary, if for all $n \in \mathbb{N}$ with $n \geqslant N$ we have

$$\rho(F^n x, x) \geqslant \epsilon,$$

then, by (3), the sequence $\{F^n x\}$ does not have any Cauchy subsequence and this is contradictory of the fact that the sequence $\{F^n x\}$ has a convergent subsequence. Therefore, for all $N \in \mathbb{N}$ fixed we have

$$\inf_{n \geqslant N} d(F^n x, x) \leqslant \inf_{n \geqslant N} \rho(F^n x, x) = 0.$$

This implies that

$$\inf_{n \geqslant N} d(F^n x, x) = 0.$$

□

Our main results in this section are the following theorems that all of them derive of Lemma 2.1.

Theorem 2.1. *Let (X, d) be a metric space and $F : X \to X$ be a ULM such that the sequence $\{F^n z\}$ has a convergent subsequence for all $z \in X$. If for all $x \in X$ there exists a function $\varphi_x : [0, +\infty) \to [0, +\infty)$ such that φ_x is continuous at 0, $\varphi_x(0) = 0$ and*

$$d(x, Fx) \leqslant \varphi_x\big(d(x, F^n x)\big) \textit{ for all } n \in \mathbb{N} \textit{ such that } F^n x \neq x, \tag{4}$$

then, there exist $n \in \mathbb{N}$ and $x \in X$ such that $F^n x = x$.

Proof. Lemma 2.1 guarantees the existence of $x \in X$ such that

$$\inf_{n \geqslant N} d(F^n x, x) = 0 \text{ for all } N \in \mathbb{N}.$$

Then there exists subsequence $\{F^{n_j} x\}$ of the sequence $\{F^n x\}$ such that $F^{n_j} x \to x$ as $j \to \infty$. If $F^n x \neq x$ for all $n \geqslant 2$, then the continuity of φ at 0 yields $\varphi_x\big(d(F^{n_j} x, x)\big) \to 0$ as $j \to \infty$. Now (4) implies that $x = Fx$ and the proof is complete. □

Theorem 2.2. *Let (X, d) be a metric space and $F : X \to X$ be a ULM such that the sequence $\{F^n z\}$ has a convergent subsequence for all $z \in X$. If for all $x \in X$ there exists a function $\varphi_x : [0, +\infty) \to [0, +\infty)$ such that φ_x is continuous at 0, $\varphi_x(0) = 0$ and*

$$d(F^n x, F^{n+1} x) \leqslant \varphi_x\big(d(x, F^n x)\big) \textit{ for all } n \in \mathbb{N}, \tag{5}$$

then, F has a fixed point.

Proof. To prove the existence of fixed point of F we use Lemma 2.1 to find $x \in X$ such that

$$\inf_{n \geqslant N} d(F^n x, x) = 0 \text{ for all } N \in \mathbb{N}.$$

Then there exists subsequence $\{F^{n_j} x\}$ of the sequence $\{F^n x\}$ such that $F^{n_j} x \to x$ as $j \to \infty$. Now (5) implies that

$$d(F^{n_j} x, F^{n_j+1} x) \leqslant \varphi_x\big(d(x, F^{n_j} x)\big) \text{ for all } j \in \mathbb{N}. \tag{6}$$

The continuity of φ at 0 yields $\varphi_x\big(d(F^{n_j} x, x)\big) \to 0$ as $j \to \infty$. Thus, by (6), we have $d(F^{n_j} x, F^{n_j+1} x) \to 0$ as $j \to \infty$. On the other hand $d(F^{n_j} x, F^{n_j+1} x) \to d(x, Fx)$ as $j \to \infty$, therefore $d(x, Fx) = 0$ as desired. □

Corollary 2.1. *Let (X, d) be a metric space and $F : X \to X$ be a ULM such that the sequence $\{F^n z\}$ has a convergent subsequence for all $z \in X$. If for all $x \in X$ there exists a constant $L_x > 0$ such that*

$$d(x, F^{n+1}x) \leqslant L_x d(x, F^n x) \text{ for all } n \in \mathbb{N},$$

then, F has a fixed point.

Proof. In view of Theorem 2.2 it is sufficient to prove that the condition (6) is satisfied. Let $x \in X$ be arbitrary. By the assumption for all $n \in \mathbb{N}$ we have

$$d(F^n x, F^{n+1}x) \leqslant d(x, F^n x) + d(x, F^{n+1}x) \leqslant (L_x + 1)d(x, F^n x).$$

Thus, by assuming that $\varphi_x(y) = (L_x + 1)y$ the proof is complete. □

The following theorem is applied in the solvability of the linear Fredholm integral equations of the second kind in the next section.

Theorem 2.3. *Let (X, d) be a metric space and $F : X \to X$ be a compact and ULM. Let there exists $y \in X$ such that the sequence $\{F^n y\}$ is bounded. If there are $x_0 \in X$, $N \in \mathbb{N}$ and $L > 0$ such that for all $m, n \in \mathbb{N}$ with $m, n \geqslant N$ we have*

$$d(F^n x_0, F^{n+m+1}x_0) \leqslant Ld(F^n x_0, F^{n+m}x_0), \tag{7}$$

then, F has a fixed point and the sequence $\{F^n x_0\}$ converges to this point.

Proof. Let F be a ULM with Lipschitzian constant L. For all $n \in \mathbb{N}$ we have

$$d(F^n y, F^n x_0) \leqslant Ld(y, x_0). \tag{8}$$

Since the sequence $\{F^n y\}$ is bounded, (8) implies that the sequence $\{F^n x_0\}$ is a bounded sequence. By the hypothesis, F is a compact map, then this sequence has a convergent subsequence. Let the subsequence $\{F^{n_j} x_0\}$ of the sequence $\{F^n x_0\}$ converges to, say, $z \in X$. We claim that z is a fixed point of F and the sequence $\{F^n x_0\}$ converges to this point. By Lemma 2.1, we can find the subsequence $\big\{d(F^{m_j} z, z)\big\}$ of the sequence $\big\{d(F^n z, z)\big\}$ such that $d(F^{m_j} z, z) \to 0$ as $n \to \infty$. Now, by (7) we have

$$d(F^{n_j} x_0, F^{n_j+m+1}x_0) \leqslant Ld(F^{n_j} x_0, F^{n_j+m}x_0) \text{ for all } j, m \geqslant N. \tag{9}$$

Taking the limit as j tends to infinity in (9) gives

$$d(z, F^{m+1}z) \leqslant Ld(z, F^m z) \text{ for all } m \geqslant N. \tag{10}$$

Now, (10) implies that

$$d(z, F^{m_j+1}z) \leqslant Ld(z, F^{m_j} z) \text{ for all } j \geqslant N. \tag{11}$$

Again, by taking the limit as j tends to infinity in (11) we conclude that $d(z, F^{m_j+1}z) \to 0$. On the other hand $d(z, F^{m_j+1}z) \to d(z, Fz)$ as $j \to \infty$, therefore $z = Fz$ as desired.

To prove the last assertion, we consider the metric ρ on X as defined in (2). We know that the map F is nonexpansive in the metric space (X, ρ). Consider the sequence $\{\rho(F^n x_0, z)\}$. For all $n \in \mathbb{N}$ we have

$$\rho(F^{n+1}x_0, z) = \rho(F^{n+1}x_0, F^{n+1}z) \leqslant \rho(F^n x_0, F^n z) = \rho(F^n x_0, z).$$

Therefore, this sequence is a decreasing sequence of real numbers that is bounded from below. Hence it is convergent to a nonnegative real number, say, a. But the sequence $\{\rho(F^{n_j} x_0, z)\}$ is a subsequence of this sequence so, is convergent to a too. On the other hand, the sequence $\{\rho(F^{n_j} x_0, z)\}$ is convergent to 0, therefore $a = 0$ and this means that $\{F^n x_0\}$ converges to z. □

Remark 2.1. *Notice that, in Theorem 2.3, if the sequence $\{F^n x\}$ is unbounded for all $x \in X$, then the map F is fixed point free.*

3. Applications

The solability of Fredholm integral equations using different techniques have been of great interest during recent years [8, 7, 14, 21]. In this section, to conclude this paper and show that our results are applicable, we consider the linear Fredholm integral equation of the second kind and prove a new existence result under suitable conditions, and then we present some illustrative examples to demonstrate the effectiveness of the obtained results.

Let μ, a and b be real numbers with $a < b$ and $\mu \neq 0$. By $C[a,b]$ we denote the normed space of all real-valued continuous functions on a given closed interval $[a,b]$ equipped with the sup-norm

$$\|f\| = \sup_{x\in[a,b]} |f(x)| \text{ for } f \in C[a,b],$$

Let $K : [a,b] \times [a,b] \longrightarrow \mathbb{R}$ be a continuous function and $V \in C[a,b]$. Now, let us consider the following linear Fredholm integral equation of the second kind

$$f(s) = V(s) + \mu \int_a^b K(s,t)f(t)dt \qquad a \leqslant s \leqslant b. \tag{12}$$

If we define the self map F on $C[a,b]$ by

$$\big(F(f)\big)(s) = V(s) + \mu \int_a^b K(s,t)f(t)dt \qquad f \in C[a,b], \tag{13}$$

then every solution of equation (13) corresponds to a fixed point of the operator F.

Define $c = \max_{a\leqslant s,t\leqslant b} |K(s,t)|$ and suppose that $c > 0$. Then by the Banach contraction principle, the linear Fredholm integral equation (13) has exactly one solution $x \in C[a,b]$ for each $\mu \in \mathbb{R}$ with

$$(b-a)|\mu|c < 1. \tag{14}$$

Also, $\{x_n\}$ converges to x from an arbitrary initial element $x_0 \in C[a,b]$ where

$$x_n(s) = V(s) + \mu \int_a^b K(s,t)x_{n-1}(t)dt,$$

and by the definition of $\|.\|$, $x_n \to x$ as $n \to \infty$, uniformly on $[a,b]$ [27].

Now, consider the sequence $\{A_n\}$ as follows

$$A_n = |\mu|^n \sup_{a\leqslant s\leqslant b} \int_\Omega \big|K(s,x_1)K(x_1,x_2)...K(x_{n-1},x_n)\big|dx, \tag{15}$$

where $\Omega = [a,b]^n$ and $x = (x_1, x_2, ..., x_n)$. It is easy to see by induction that

$$\|F^n(f) - F^n(g)\| \leqslant A_n \|f - g\| \text{ for all } f, g \in C[a,b] \text{ and } n \in \mathbb{N}. \tag{16}$$

Where F is defined in (14).

The main result of this section is the following theorem.

Theorem 3.1. *The linear Fredholm integral equation* (13) *has a solution if the following conditions hold*

(1) $\sup_{n\in\mathbb{N}} A_n < \infty$,

(2) $\sup_{n\in\mathbb{N}} \|F^n(0)\| < \infty$,

(3) *there exist* $f \in C[a,b]$, $N \in \mathbb{N}$ *and* $L > 0$ *such that for all* $m, n \in \mathbb{N}$ *with* $m, n > N$ *we have*

$$\|F^n(f) - F^{n+m+1}(f)\| \leqslant L\|F^n(f) - F^{n+m}(f)\|,$$

where, the sequence $\{A_n\}$ *and the map* F *are defined in* (15) *and* (13)*, respectively. Furthermore, the Picard iteration* $\{F^n(f)\}$ *converges to this solution.*

Proof. By using Arzela-Ascoli's theorem, it can be easily showed that the map F (defined in (13)) is compact (see for example [19]). On the other hand, by (1) and (16) the map F is a ULM, then by Theorem 2.3 the map F has a fixed point and the Picard iteration $\{F^n(f)\}$ converges to this fixed point. $\square$

Example 3.1. *Consider the following linear Fredholm integral equation of the second kind:*

$$f(s) = V(s) + \frac{4}{\pi+2}\int_0^{\frac{\pi}{2}} \sin(s+t)f(t)dt, \tag{17}$$

where $V \in C\big[0, \dfrac{\pi}{2}\big]$ *is a given function such that*

$$\int_0^{\frac{\pi}{2}} \big(\sin(t) + \cos(t)\big)V(t)dt = 0. \tag{18}$$

Here we have $\mu = \frac{4}{\pi+2}$ *and* $K(s,t) = \sin(s+t)$*. It can be easily checked that if* $a, b \in \mathbb{R}$ *are arbitrary, then*

$$\frac{4}{\pi+2}\int_0^{\frac{\pi}{2}} \sin(s+t)\big(a\sin(t) + b\cos(t)\big)dt = \frac{b\pi+2a}{\pi+2}\sin(s) + \frac{a\pi+2b}{\pi+2}\cos(s). \tag{19}$$

Let the sequence $\{A_n\}$ *be as defined in* (15)*. It is easy to see by induction and* (19) *that for all* $n \in \mathbb{N}$ *we have*

$$\Big(\frac{4}{\pi+2}\Big)^n \int_{[0,\frac{\pi}{2}]^n} \sin(s+x_1)...\sin(x_{n-1}+x_n)dx_n...dx_1 = \frac{4}{\pi+2}\big(\sin(s) + \cos(s)\big).$$

Hence, $\sup_{n\in\mathbb{N}} A_n = \dfrac{4\sqrt{2}}{\pi+2} > 1$*. Now, consider the sequence* $\{F^n(0)\}$ *where* F *is defined in* (13)*. If we define* $\alpha = \displaystyle\int_0^{\frac{\pi}{2}} \cos(t)V(t)dt$*, then it is not hard to see by induction on* n*,* (18) *and* (19) *that*

$$\big(F^n(0)\big)(s) = V(s) + \frac{4\alpha}{\pi+2}\sum_{k=0}^{n-2}\Big(\frac{2-\pi}{2+\pi}\Big)^k\big[\sin(s) - \cos(s)\big] \ \textit{for all}\ n \geqslant 2. \tag{20}$$

Thus, $\sup\limits_{n\in\mathbb{N}}\|F^n(0)\| \leqslant |\alpha| + \|V\|$. *By* (20) *we conclude that*

$$\Big|\big(F^n(0)\big)(s) - \big(F^{n+m+1}(0)\big)(s)\Big| = \frac{\Big|\sum\limits_{k=n-1}^{n+m-1}(\dfrac{2-\pi}{2+\pi})^k\Big|}{\Big|\sum\limits_{k=n-1}^{n+m-2}(\dfrac{2-\pi}{2+\pi})^k\Big|}\Big|\big(F^n(0)\big)(s) - \big(F^{n+m}(0)\big)(s)\Big| =$$
$$= \frac{1-(\frac{2-\pi}{2+\pi})^{m+1}}{1-(\frac{2-\pi}{2+\pi})^{m}}\big|\big(F^n(0)\big)(s) - \big(F^{n+m}(0)\big)(s)\big| \leq$$
$$\leqslant 2\big|\big(F^n(0)\big)(s) - \big(F^{n+m}(0)\big)(s)\big|.$$

Thus

$$\|F^n(0) - F^{n+m+1}(0)\| \leqslant 2\|F^n(0) - F^{n+m}(0)\| \; for\; all\, n, m \geqslant 2.$$

Therefore, all the conditions in Theorem 3.1 *are satisfied. Thus, the Picard iteration* $\{F^n(0)\}$ *converges to a solution of the linear Fredholm integral equation* (17). *By* (20) *we have*

$$\big|\big(F^n(0)\big)(s) - V(s) - 2\alpha\big(\sin(s) - \cos(s)\big)\big| \to 0 \; as\; n \to \infty.$$

Then $V(s) + 2\alpha\big(\sin(s) - \cos(s)\big)$ *is a solution of the linear Fredholm integral equation* (17). *Notice that the previous result can not be applied here since* $|\mu|c(b-a) = \frac{2\pi}{2+\pi} > 1$. *Hence,* (14) *does not hold.*

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